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Variational Data Assimilation for Hamiltonian Problems

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NUMERICAL ANALYSIS REPORT 4/04

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Paper presented at ICFD conference on Numerical Methods for Fluid Dynamics, Oxford, UK,
29 March - 1 April 2004.

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Abstract

We investigate the conservation properties of Hamiltonian systems in variational data assimilation. We set up a four dimensional data assimilation scheme for the two-body (Kepler) system using a symplectic scheme to model the non-linear problem. We use our completed scheme to investigate the observability of the system and the effect of different background constraints. We find that the addition of these constraints gives an improved solution for the cases we have investigated.

1 Introduction

Some features of atmospheric dynamics can be modelled using Hamiltonian methods. We investigate whether the conservation properties of such systems can be exploited when using data assimilation schemes. To do this we set up a full 4d variational (4D-Var) data assimilation scheme for the simpler problem of planetary orbits, which also has a Hamiltonian structure. The first section introduces variational data assimilation. We then discuss the two-body problem in its continuous and discrete form. Section 3 shows the results of our assimilation experiments.

1.1 4D Variational Data Assimilation

information at the initial time. In the first case we constrain the background model state vector directly. In the second case this information is transformed to an energy thus using the Hamiltonian property as a constraint. For the first of these we add to the cost function a term of the form $\mathbb{R}_1(\mathbf{x}_{b_0} - \mathbf{x}_0)(\mathbf{x}_{b_0} - \mathbf{x}_0)^T$, for the second the constraint term is $\mathbb{R}_2(E(\mathbf{x}_{b_0}) - E(\mathbf{x}_0))^2$. Here $\mathbb{R}_1$ and $\mathbb{R}_2$ are parameters that allow the effect of each of the terms to be controlled.

2 Modelling the Two-Body Problem

2.1 The Continuous Problem

The two-body problem is one of the simplest Hamiltonian problems. Instead of writing the system as two particles of mass m_1 and m_2 in mutual orbit, we can set the origin at the centre of mass. This reduces the problem to one particle of reduced mass 1 , where $^1 = \frac{m_1 m_2}{m_1 + m_2}$, in orbit around one particle of total mass, $m_1 + m_2$. Our continuous equations of motion can therefore be written as two first order, non-dimensional equations describing the evolution of position, $\mathbf{q} = (q_1; q_2)$, and momentum, $\mathbf{p} = (p_1; p_2)$,

$$\frac{d\mathbf{q}}{dt} = \mathbf{p} \quad \frac{d\mathbf{p}}{dt} = -\frac{\mathbf{q}}{(q_1^2 + q_2^2)^{\frac{3}{2}}}. \quad (2)$$

2.1.1 Conservation Properties

The two-body problem has two conserved quantities, the Hamiltonian, E , which for this problem is the total energy, and the angular momentum, L . These are given by,

$$E = \frac{1}{2} p_1^2 + p_2^2 - \frac{1}{(q_1^2 + q_2^2)^{\frac{1}{2}}} = \text{constant} \quad L = q_1 p_2 - p_1 q_2 = \text{constant:} \quad (3)$$

These characteristics are intrinsic to the physical problem, and will provide a useful test of the discretised equations.

2.2 The Discrete Problem

To test the effect of these conservation properties in the 4D-Var scheme it is essential that they are captured by the discrete model. In recent years *geometric integration* has

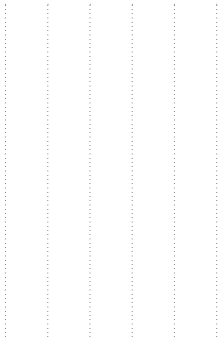


Figure 1: Difference between the energy given by the model and the true energy for (a) eccentricity, $e = 0$, and (b) $e = 0.5$

attempted to address the issue of preserving global features, and in particular *symplectic methods* are particularly good at conserving energy, and can also conserve angular momentum [3]. Following previous work on the two-body problem we use a second order, symplectic, Runge-Kutta scheme known as the Störmer-Verlet method [4].

Figures 1(a) and 1(b) illustrate how well the model captures the energy conservation for a circle (where eccentricity, $e = 0$) and an ellipse with $e = 0.5$. These figures show the difference between the energy given by the model and the truth. For a circle this difference is at a scale of 10^{-14} , and so here the scheme does well. However we see that if

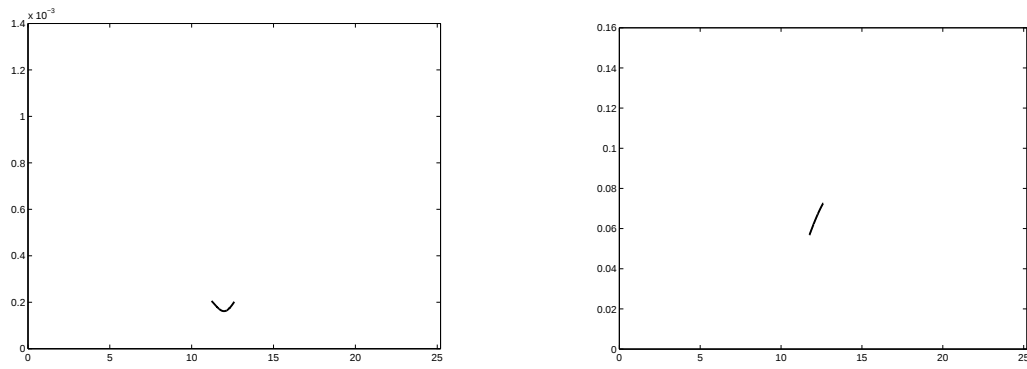


Figure 2: Error between the optimal solution and the truth for $e = 0$ and assimilation window $t = 12:6$, for (a) dense observations, and (b) sparse observations

the solution from the given observations. This limited analysis suggests that the solution from the given observations is not accurate.

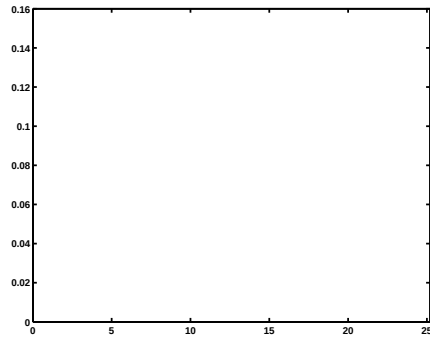


Figure 3: Error between the optimal solution and the truth using sparse observations of momentum with (a) a background constraint, and (b) an energy constraint ($e = 0$, assimilation window $t = 12:6$)

3.2.1 Constraint using x_b

Here we alter the cost function and its gradient to include a background term. This means

Acknowledgements

I would like to thank Prof M. Baines and Prof M. Cullen for their help with this research.

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