The University of Reading

Approximate Gauss-Newton

be the model state ve[redacted]vector of p_j observations and $h_j : \mathbb{R}^n \rightarrow \mathbb{R}^{p_j}$ be a[redacted] relates the system states to the observations at ti[redacted]ation problem is then de¯ned as follows.

Problem 1. [redacted] x_0, *the objective function*

$\mathcal{J}[x_0] = \frac{1}{}$ [redacted] $) + \frac{1}{2} \sum_{j=0}^{N} (h_j(x_j) - y_j)^T R_j^{-1} (h_j(x_j) - y_j);$ (5)

subj[redacted] *satisfying the discrete dynamical system equa-* = 0r0TD (that1: 96p)]TJ/lyTD[(fotor,)]TJ0torop((to) 57ned

tio[redacted]

T[redacted]e, x_0^b 0, *the*

Low dimensional system models can be obtained by using low resolution approximations to the full dynamical system. Significant features of the system behaviour are often lost, however, in such approximations. In particular optimal error growth modes may not be captured by these models. In the next section we propose an alternative method for generating low order system approximations using techniques of model reduction.

3 MODEL REDUCTION BY BALANCED - TRUNCATION

To find low order approximations to the linearized system model (6), we project the system into a low dimensional subspace. We introduce linear restriction operators $U_j^T \in \mathbb{R}^{r \times n}$ that project the state variables into the subspace $\mathbb{R}^r$ where $r << n$. We define variables $\delta\hat{x}_j \in \mathbb{R}^r$, such that $\delta\hat{x}_j = U_j^T \delta x_j$, and define prolongation operators $V_j \in \mathbb{R}^{n \times r}$ that project the variables back into the original space $\mathbb{R}^n$. The restriction and prolongation operators U_j^T and V_j satisfy $U_j^T V_j = I_r$ and $V_j U_j^T$ is a projection operator. We write the projected linear system as

$$\delta\hat{x}_{j+1} = U_j^T M_j V_j \delta\hat{x}_j; \qquad \hat{d}_j = H_j V_j \delta\hat{x}_j. \tag{8}$$

The reduced-dimension inner minimization problem then becomes

Problem 3. *Minimize, with respect to $\delta\hat{x}_0^{(k)}$, the objective function*

$$\begin{aligned} \hat{\mathcal{J}}^{(k)}[\delta\hat{x}_0^{(k)}] &= \frac{1}{2}(\delta\hat{x}_0^{(k)} - U_0^T[x^b - x_0^{(k)}])^T \hat{B}_0^{-1}(\delta\hat{x}_0^{(k)} - U_0^T[x^b - x_0^{(k)}]) \\ &+ \frac{1}{2}\sum_{i=0}^{N}(H_j V_j \delta\hat{x}_j^{(k)} - d_j^{(k)})^T R_j^{-1}(H_j V_j \delta\hat{x}_j^{(k)} - d_j^{(k)}); \end{aligned} \tag{9}$$

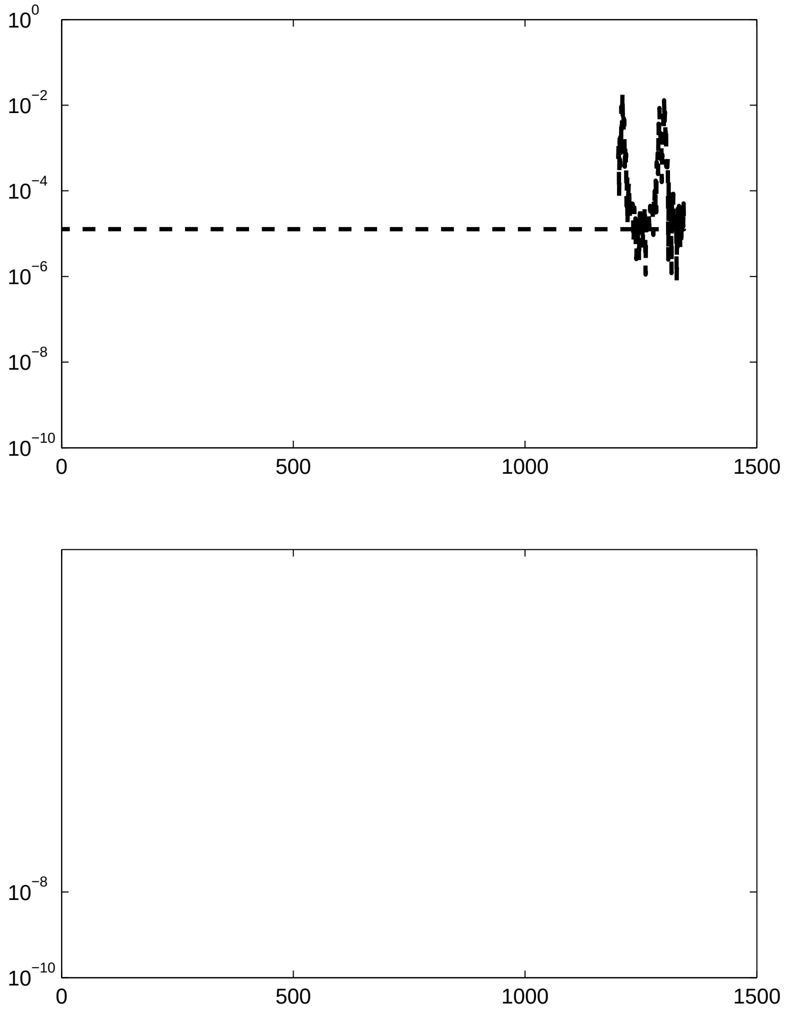

Figure 1: Errors in solutions to reduced linear least squares problem for low resolution model (dotted) of order $r = 750$ and optimal reduced order model (dashed) of order (a) $r = 750$ and (b) $r = 250$.

r = 750, the errors between the exact solution and the solutions obtained using the two di®erent low dimensional models are shown in Figure 1(a). The least square norms of the errors in the two cases are given, respectively, by (i) 0:0396 and (ii) 0:0057. It is clear that for the same model size, the optimal reduced order models are signi¯cantly more accurate than the low resolution models. This bene¯t can be explained in part by examining the eigenstructure of the reduced dimensional systems. More of the signi¯cant eigenvalues of the optimal reduced order model match those of the full system than is the case for the low resolution model, showing that the modes of the system are more accurately captured by the balanced-truncation method than by using low resolution models.

Solutions to the reduced least squares problem obtained for smaller values of r demonstrate that balanced-truncation can be applied to ¯nd much smaller systems with accuracy equal to that of the low resolution model. A

[3] Courtier, P., Thépaut, J-N. and Hollingsworth, A. A strategy for operational implementation of 4D-Var, using an incremental approach, *Q. J. R. Meteorol. Soc.*, **120**, 1367{1387, 1994.

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