M A S T E R F U L

Monotonic Adaptive Solutions of Transient Equations using Recovery, Fitting, Upwinding and Limiters

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Abstract

A method of approximately solving transient, particularly convective, equations on adaptive grids is presented, which has the useful properties (i) that both the grid and the solution remain monotonic, and (ii) that the solution is always the best fit with adjustable nodes to a recovered smoother version of itself. In this report the underlying representation is piecewise constant and the recovered function piecewise linear, while the equation is the inviscid Burgers' equation solved by upwind finite differences, but other representations, equations and schemes are also discussed.

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1 Introduction

The solutions of time-dependent partial differential equations may exhibit steep fronts whose locations vary with time and are difficult to predict in advance. Resolution of the resulting profiles can be achieved by a fine grid everywhere, but such a grid is impractically large, especially for 3-D calculations.

An adaptive grid is therefore demanded, one which will ideally follow the fronts around and provide the required resolution in a selective way. In order to achieve this ability, however, two difficult questions must be addressed. The first is the problem of representation of the solution on an irregular grid or rather, in this context, the problem of knowing the best irregular grid to represent the solution. The second is the control of the grid as the solution changes position and shape, in particular the prevention of node overtaking (which is the equivalent in nodal terms of monotonicity in the solution profile).

The first problem has been tackled by a number of authors in the context of approximation theory, in particular de Boor [1], Chui [2], Loach Wathen [3], Farmer, Heath Moody [4] and Baines [5]. The latter gives algorithms for obtaining best L_2 fits to continuous functions using piecewise constant and piecewise linear representations with adjustable nodes which are relatively simple and robust and are the basis of the nodal movement used here. Of course in the present context there is no given function to be fitted, since the exact solution of the PD is not known, but we get round this difficulty by predicting the new numerical solution and makinguse of a recovered function (see below).

The second problem, that of nodal movement and its control, has also been the subject of many papers, in particular Dorfi Drury [6], Petzold [7] and Miller [8] (see also Verwer et al. [9]). The moving finite element (MF) method method of Miller [8] is an attempt to move the nodes by the same mechanism which controls the solution, namely, consistency with the underlying PD . As shown by Baines [10], the result for first order PD 's is a characteristic0following method (akin to a Hamiltonian approach), but in the case of higher order PD 's the nodal velocities generated are less well understood and their effectiveness is more dubious, except in the steady state limit (Jimack [13]). In any case Miller [8]

solving PD 's with convective terms it is usually necessary to employ some kind of upwind differencing to achieve monotonicity. By going for monotonicity (both in the solution and the grid) the numerical viscosity generated provides the usual non-overturned shock-like approximation to the solution of a convective equation and its corresponding jump condition, but will be less than for a fixed grid.

In the next section we concentrate on a particular discrete solution representation and work through the method proposed above for a particular equation, later indicating the possible generalisations and their properties.

2 Piecewise onstants in 1-D

Suppose that the underlying discrete representation of the solution of a PD is piecewise constant, as shown in Fig. 1.

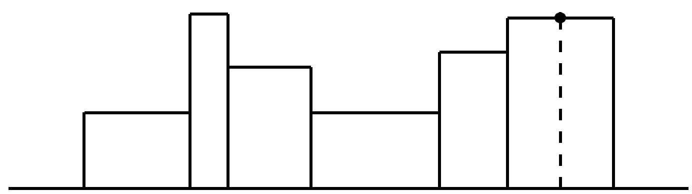

Figure 1:

Individual point values are thought of as existing at the midpoints of each element as in the rightmost cell of Fig. 1. We consider here the Cauchy problem for the inviscid Burgers' equation problem

$$u_t + uu_x = 0 \quad (t > 0) \tag{2.1}$$

$$u = u_0(x) \quad (t = 0) \tag{2.2}$$

A non-adaptive scheme for this problem is the upwind finite difference method

$$U_j^{n+1} = U_j^n - |U_j^n| \frac{\Delta t}{\Delta_o X_j^n}(U_j^n - U_{j-\mu}^n) \tag{2.3}$$

where $\mu = \text{sgn}(U_j^n)$ and $\Delta_o X_j^n = \frac{1}{2}(X_{j+1}^n - X_{j-1}^n)$. (The right hand side of (2.3) represents a linear interpolation of the values of U_j^n after being traced back along the characteristics.) While consistent with the differential equation (2.1), the scheme (2.3) is highly diffusive and, particularly in the case of a steepening wave, gives very poor representation of the exact solution. The philosophy of adaptive grids is that sharper shocks can be achieved by clustering of the grid points in a controlled way, namely, in the vicinity of the shock, and for this we use the grid movement strategy described earlier.

The first task, however, is to represent the function $u_0(x)$ of (2.2) as well as possible using piecewise constant functions on a grid with free nodes. This problem is addressed by Baines [5], where a simple algorithm is given to generate such a representation. In summary, this is as follows. For a given continuous function $f(x)$, an initial arbitrary grid is set up in each element of which a constant approximation is found by a local L_2 projection (Fig. 2).

Figure 2:

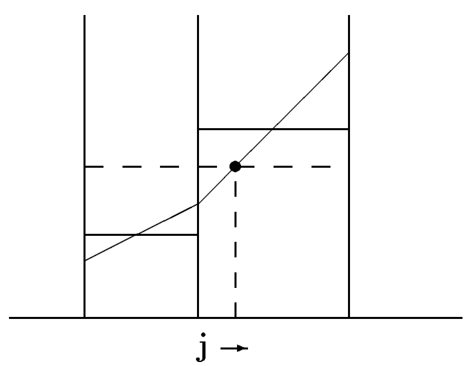

Figure 3:

Then, taking two adjacent elements, the averaged constant value is calculated (shown dotted in Fig. 2) and its intersection with the $f(x)$ function found. All grid points are treated in this way in a sweep, before another sweep is done. At convergence the best representation is obtained. A key property of the construction is that the nodes remain ordered, i.e. there is no node tangling.

The procedure also achieves the same result when $f(x)$ is replaced by the piecewise linear interpolant function, with values $f(X_j)$ at X_j as in Fig. 3. This is important for what follows.

Having generated a best initial grid and profile, the next task is to try and maintain this property as the solution evolves. As is well known, an application of the algorithm (2.3) on a fixed grid results in a profile of the correct general shape, but diffused. We wish to use this (piecewise constant) profile to determine a suitable grid movement. The best fit algorithm of Baines [5] operates on a continuous function (or a piecewise linear continuous function), which we do not have. But we can construct a piecewise linear recovered function from the new piecewise constant function, which is shown in Fig. 4 as the piecewise linear function linking the mid-points of the piecewise constant cell values.

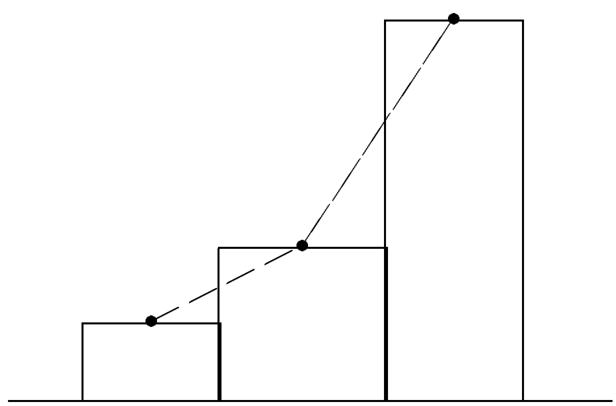

Figure 4:

As explained above, the algorithm for finding the best piecewise constant fit to this piecewise linear function is in the form of an iteration, the first step of which gives a new grid position X_j, which may be regarded as being carried out by a certain grid movement $\dot{X}_j$. The formula for the new grid position X_j is easily obtained, being the intersection between the piecewise linear function and

$$- \quad - \quad {}^{n}_{j+1} \quad {}^{n}_{j} \quad - \quad {}^{n}_{j} \quad {}^{n}_{j \ 1} \quad {}^{n}_{j} \quad - \quad {}^{2} \quad {}_{j}$$

$$
{}^{n+1}_{j} \quad
\begin{matrix}
{}^{n}_{j} \quad \frac{1}{4}\frac{\delta^2 U_j^n}{M^n_{j-\frac{1}{2}}} \quad {}^{n}_{j} \quad {}^{n}_{j \ 1} \quad {}^{n}_{j+1} \quad {}^{n}_{j} \\
{}^{n}_{j} \quad \frac{1}{4}\frac{\delta^2 U_j^n}{M^n_{j+\frac{1}{2}}} \quad {}^{n}_{j} \quad {}^{n}_{j \ 1} \quad {}^{n}_{j+1} \quad {}^{n}_{j}
\end{matrix}
$$

$${}^{n}_{j \ \frac{1}{2}} \quad \frac{{}^{n}_{j} \quad {}^{n}_{j \ 1}}{{}^{n}_{j} \quad {}^{n}_{j \ 1}} \qquad {}^{n}_{j+\frac{1}{2}} \quad \frac{{}^{n}_{j+1} \quad {}^{n}_{j}}{{}^{n}_{j+1} \quad {}^{n}_{j}}$$

$x \qquad x$

${}^{n}_{j}$

$${}^{n+1}_{j} \quad {}^{n}_{j} \quad {}^{n}_{j} \quad {}^{(1)}_{j} \quad \frac{\quad}{o \quad {}^{n}_{j}} \quad {}^{n}_{j} \quad {}^{n}_{j} \ \mu$$

${}^{n}_{j} \quad {}^{(1)}_{j} \qquad {}^{(1)}_{j}$

$${}^{(1)}_{j} \quad \frac{{}^{(1)}_{j} \quad {}_{j}}{\quad}$$

${}^{(1)}_{j} \quad {}_{j}$

x

$$
{}^{n+1}_{j} \quad
\begin{matrix}
{}^{n}_{j} \quad \frac{1}{4}\frac{\delta^2 U_j}{M^n_{j-\frac{1}{2}}} \quad {}^{n}_{j} \quad {}^{n}_{j \ 1} \quad {}^{n}_{j+1} \quad {}^{n}_{j} \\
{}^{n}_{j} \quad \frac{1}{4}\frac{\delta^2 U_j}{M^n_{j+\frac{1}{2}}} \quad {}^{n}_{j} \quad {}^{n}_{j \ 1} \quad {}^{n}_{j+1} \quad {}^{n}_{j}
\end{matrix}
$$

$$+1 \qquad \frac{+1}{\quad} \quad \frac{\quad}{\quad}$$

where

$$\mu = \quad U_j^n \quad \frac{(X_j^{n+1} \quad X_j^n)}{\Delta t} \quad . \tag{2.12}$$

The result is a piecewise constant profile at time level $n+1$, obtained by a finite difference scheme consistent with the PD , which is a best L_2 fit to the piecewise linear recovered function (standing in lieu of the exact solution) at time level $n+1$.

The algorithm of section 2 is

$$X_j^{n+1} = \begin{matrix} X_j^n + \frac{1}{4}\frac{\delta^2 U_j^n}{M_{j-\frac{1}{2}}^n} \quad U_j^n \quad U_{j-1}^n \quad {}_{j+1}^{n} \quad {}_j^n \quad U \\ {}_j^n + \frac{1}{4}\frac{\delta^2 U_j^n}{M_{j+\frac{1}{2}}^n} \quad {}_j^n \quad {}_{j-1}^n \quad {}_{j+1}^n \quad {}_j^n \end{matrix} \tag{3 1}$$

$${}_j^{n+1} = {}_j^n \quad {}_j^n \quad \frac{({}_j^{n+1} \quad {}_j^n)}{\Delta} \quad \frac{\Delta}{\Delta_o \; {}_j^n}({}_j^n \quad {}_{j-\mu}^n) \tag{3 2}$$

where

$$= \mathrm{sgn} \quad {}_j^n \quad \frac{(X_j^{n+1} - X_j^n)}{\Delta t} \qquad {}_{j-\frac{1}{2}}^n = \frac{U_j^n - U_{j-1}^n}{X_j^n - X_{j-1}^n}$$

$$\Delta_o \; {}_j^n = \tfrac{1}{2}({}_{j+1}^n \quad {}_{j-1}^n) \tag{3 3}$$

The geometrical construction of ${}_j^{n+1}$ ensures that

$${}_j^n \quad {}_j^{n+1} \quad \frac{1}{2}({}_j^n + {}_{j+1}^n) \tag{3 4}$$

for

$${}_{j-\frac{1}{2}}^n \quad {}_{j+\frac{1}{2}}^n \quad {}_{j+1}^n \quad {}_j^n \quad {}_j^n \quad {}_{j-1}^n$$

and

$$\frac{1}{2}({}_{j-1}^n + {}_j^n) \quad {}_j^{n+1} \quad {}_j^n \tag{3 5}$$

for

$${}_{j-\frac{1}{2}}^n \quad {}_{j+\frac{1}{2}}^n \quad {}_j^n \quad {}_{j-1}^n \quad {}_{j+1}^n \quad {}_j^n$$

These inequalities can also be proved analytically from (3.1). No tangling is therefore possible and monotonicity of the mesh is achieved. Moreover the upwinded nature of the scheme (3.2) means that, provided that a CFL limit is respected, monotonicity of $_j$ is also preserved. The CFL restriction is that

$$ {}^{n}_{j} \quad \frac{({}^{n+1}_{j} \qquad {}^{n}_{j})}{\Delta} \quad \frac{\Delta}{{}_{o} \; {}^{n}_{j}} $$

$$ 0 \qquad\qquad - $$

$$ {}^{n+1}_{j} \qquad {}^{n}_{j} \qquad {}^{n}_{j-\frac{\mu}{2}} \frac{}{{}_{o} \; {}^{n}_{j}} \qquad {}^{n}_{j} \qquad {}^{n}_{j-\mu} $$

$$ {}^{n}_{j-\frac{\mu}{2}} \qquad - \qquad {}^{n}_{j} \qquad {}^{n}_{j-\mu} $$

$$ {}^{n}_{j-\frac{\mu}{2}} $$

$$ {}^{n+1}_{j} \qquad {}^{n}_{j} \qquad \frac{}{{}_{o} \; {}^{n}_{j}} \quad - \quad {}^{n}_{j} \, {}^{2} \quad - \quad {}^{n}_{j-\mu} \, {}^{2} $$

$$ {}^{n+1}_{j} \qquad {}^{n}_{j} \qquad \frac{}{{}_{o} \; {}^{n}_{j}} \quad - \quad {}^{n}_{j} \, {}^{2} \quad - \quad {}^{n}_{j-\mu} \, {}^{2} $$

$t \quad x \quad - \quad \frac{\quad 2}{0} \quad xx \quad \frac{2}{0} \quad x \quad 2 \quad 2$

$j \quad j \quad 1 \quad 0 \quad \frac{1}{2} \quad j+1 \quad j \quad 1 \quad 2 \quad j+1 \quad j \quad j \quad 1$

$\frac{}{2 \quad 0}$

$- \quad \frac{\quad 2}{0} \quad \frac{}{} \quad \frac{2}{0} \quad 2 \quad 2$

$- \quad \frac{\quad 3}{0} \quad \frac{2}{} \quad \frac{2}{0}$

1

$+1 \quad - \quad \frac{}{0}^{2} \quad \frac{2}{\frac{1}{2}} \quad \frac{}{0}^{2} \quad 2$

$+1 \quad - \quad \frac{+}{0}^{2} \quad + \quad + \quad \frac{2}{+\frac{1}{2}} \quad \frac{+}{0}^{2} \quad + \quad 2$

$+ \quad +1$

$\frac{1}{2}$

$\frac{1}{8}$

of piecewise linears can be achieved using limited Hermite cubics as in Priestly [21] and all the ingredients of the method are then in place. The piecewise linear best fit algorithm of Baines [5] again involves no tangling of the grid, but this time the movement of the grid is towards greater curvature in the profile rather than greater steepness (Baines [5]).

While the method presented here is directed mainly at convective equations where node clustering is required near fronts, there is no reason why other equations cannot be treated by the same method.

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