Plane Wave Discontinuous Galerkin Methods: Exponential Convergence of the hp-version

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Abstract

We consider the two-dimensional Helmholtz equatT43857(55(e2a3.I8(e)1.9245t1323p7T8(e)1.924m37a)4.43552(t)4
representative is the so-called Ultra Weak Variational Formulation (UWVF), proposed in [

5].

element methods. Assuming domains and data with sufficient regularity, the idea is to use large mesh cells equipped with many plane waves where the solution is smooth, whereas small cells are employed to resolve singularities of the solution at corners of the boundary. This kind of hp approximation with polynomials has seen an amazing development starting from the work of Babuška [1, 10]; see [32] for a comprehensive exposition. It has also been adapted to polynomial DG methods by several authors, see, for instance, [16, 30, 31, 35]. Applications to scalar wave propagation are reported in [7, 23, 24].

Results on the approximation of Helmholtz solutions by plane waves are pivotal. Here, major progress has been achieved in [26, 27]. These works made use of Vekua's theory and, thus, could exploit known results about the approximation of harmonic functions by harmonic polynomials. Recently, results in this direction targeting harmonic functions that can be extended analytically were obtained in [15], generalising earlier work by M. Melenk [20]. A proof of exponential convergence of the hp-version of (polynomial) Trefftz-DG method for the Laplace problem was included.

The main result of this work (Theorem 6.5, Section 6) is a proof that the L^2-norm of the discretisation error of a special PWDG method on very general, geometrically graded meshes converges exponentially in a root of the number of degrees of freedom. This is the first such result for a numerical method based on plane waves. For the proof, we had to refine the duality arguments of [14], see Section 4, and combine them with novel L^∞-approximation estimates for plane waves given in Section 5. The reason of the restriction to two space dimensions is that the approximation estimates for harmonic functions we rely on (see Proposition 5.1) were derived in [15] using complex analysis arguments, and thus are proved in 2D only. The error is bounded by a negative exponential of the square root of the total number of degrees of freedom employed, while typical polynomial hp-schemes in two dimensions only deliver exponential convergence in the cubic root of the same parameter, e.g. see [1, Theorem 5.3]. The results of our analysis hold true also when circular waves are used instead of plane waves.

At this point we emphasise that our focus is on numerical approximation theory. We deliberately ignore the key challenge of ill-conditioning of linear systems arising from PWDG approaches, cf. [17, 18]. We even acknowledge that an implementation of the method investigated below may severely be affected by numerical instability, see Remark 6.7.

2 Scattering boundary value problem

As in [14, Section 2], let $\Omega_D \subset \mathbb{R}^2$ be a bounded, Lipschitz domain occupied by a sound-soft material, which we assume to be star-shaped with respect to the origin 0. We denote by $\Gamma_D := \partial\Omega_D$ its boundary. We introduce another bounded Lipschitz domain Ω_R with boundary Γ_R such that $\overline{\Omega}_D \subset \Omega_R$, and $\mathrm{dist}(\Gamma_D; \Gamma_R) > 0$[1]. We set $\Omega := \Omega_R \setminus \overline{\Omega}_D$ and we assume $\partial\Omega$ to be piecewise analytic. It may have finitely many corners c_ℓ, $1 \le \ell \le n_c$, which we collect in the set $C := \{c_\ell\}_{\ell=1}^{n_c}$.

We focus on the following boundary value problem (BVP) for the Helmholtz equation:

$$\begin{cases} -\Delta u - k^2 u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ \nabla u \cdot \mathbf{n} + ik\vartheta u = g_R & \text{on } \Gamma_R, \end{cases} \tag{1}$$

with $g_R \in L^2(\Gamma_R)$, wavenumber $k > 0$, and $\vartheta \in \mathbb{R}$ a non-dimensional, non-zero parameter. We have written $\mathbf{n}$ for the outward-pointing unit normal vector field on $\partial\Omega$.

2.1 Stability and Sobolev regularity

We denote by

norms (note that k has the dimension of the inverse of a length):

$$\|v\|^2_{\ell;k;D} := \sum_{j=0}^{\ell} k^{2(\ell - j)} |u|^2_{j;D} \qquad \forall v \in H^{\ell}(D);\ \ell \in \mathbb{N}.$$

We assume $_R$ to be star-shaped with respect to the ball[2] $B_{_R d}$, for some $_R > 0$, where d := diam( ).

Theorems 2.1, 2.2, and 2.3 of [

where

$$_{p;\,;\,E}(x) = \min\left\{1; \frac{|x|}{\min\{1; E(|p|+1)\}}\right\}^{p+} :$$

We set $b(x) := b_{1;\underline{0};1}(x) = \prod_{=1}^{n_c} \min\{1; |x - x_c|\}$, which is independent of k.

Theorem 2.3. There exists a weight vector $\underline{\ } \in (0;1)^{n_c}$ such that, if $g_R \in B^1_{\underline{\ };E}(\ _R)$, the solution u to problem (

4.3 Trace inequalities

As technical tools we use the following trace inequalities:

$$\|v\|_{0;\partial K}^2 \leq C_1 \left(h_K^{-1} \|v\|_{0;K}^2 + h_K |v|_{1;K}^2 \right) \qquad \forall v \in H^1$$

$$C \sum_{K \in T_h} \| \, ^{\frac{1}{2}} \|^2_{L^1(\partial K \cap (F_h^I \cup \; _D))} \left(\frac{1}{k h_K} \| r \; v \|^2_{0;K} + \frac{h_K^{2s}}{k} | r \; v |^2_{\frac{1}{2}+s;K} \right) ;$$

with $C >$

and thus, due to assumption (M3),

$$\frac{|(w;\ )_{0;}\ |}{\|\ \|_{0;}}\quad C\,\frac{(C_F + |\ _R|)d^2}{|\ |}\quad \frac{1}{kh_{max}} + d\ k + (d\ k)^3\quad \|w\|_{DG}\ ;$$

and the result readily follows. □

Since $u \quad u_{hp}$ 2 $T(T_h)$, from Lemma 4.4 and the quasi-optimality (8), we immediately deduce the following result.

Theorem 4.5. Assume the mesh properties (M1){(M3) and that the solution u of (1) belongs to $T(T_h)$, and let u_{hp} be the solution of (7). Then there exists a constant $C > 0$ depending only on the shape of , #, $_0$, , s, a, and o5,

The whole proof is just a modi cation of those in Sections 3.4.2 and 3.5 fo[

The norm of the harmonic polynomial $V_2[Q_N]$ is immediately controlled by that of u using the triangle inequality and recalling the definition of Q_N:

$$\begin{aligned}\|V_2[Q_N]\|_{L^1(K)} &\le \|V_2[u]\|_{L^1(K)} + \|V_2[u] - V_2[Q_n]\|_{L^1(K)} \\ &\le C\left(\|V_2[u]\|_{L^1(K)} + h_K \|\nabla V_2[u]\|_{L^1(K)}\right)\end{aligned} \tag{23}$$

Assumption 6.1. Let $0 < \sigma < 1$ be a fixed grading parameter. The elements of every mesh $\mathcal{T}_L$ can be grouped into layers $\mathcal{L}^L_\ell$, $1 \le \ell \le L$, that is,

$$\mathcal{T}_L = \bigcup_{\ell=1}^{L} \mathcal{L}^L_\ell; \qquad \mathcal{L}^L_\ell \cap \mathcal{L}^L_{\ell'} = \emptyset \ \text{ if } \ell \neq \ell',$$

such that:

(GM1) the Lth layer $\mathcal{L}^L_L$ contains the set of elements abutting a corner;

(GM2) the distance of an element from the nearest corner point depends geometrically on its layer index (recalling that $\mathcal{C} = \{c_\gamma\}_{\gamma=1}^{n_c}$ is the set of corner points):

$$\exists C > 0: \quad C^{-1}\sigma^{\ell} \le \mathrm{dist}(K, \mathcal{C}) \le C\sigma^{\ell} \quad \forall K \in \mathcal{L}^L_\ell; \quad 1 \le \ell < L; \quad L \in \mathbb{N}; \tag{33}$$

(GM3) the size of an element depends geometrically on its layer index:

$$\exists C > 0: \quad C^{-1}\sigma^{\ell} \le h_K \le C\sigma^{\ell} \quad \forall K \in \mathcal{L}^L_\ell; \quad 1 \le \ell \le L \in$$

with d e selecting the smallest integer greater than or equal to L^{1+}. The role of is explained in Section 6.5. For the sake of simplicity, we opt for equi-spaced plane wave directions (i.e., = 1 in Proposition 5.4)

$$d_m^p = \begin{pmatrix} \cos(\frac{2}{p}m) \\ \sin(\frac{2}{p}m) \end{pmatrix}; \quad 0 \le m < p; \quad p \in N;$$

which give rise to the local plane wave spaces

$$PW_{p;k}(K) := \Big\{ v \in C^1(R^2) : v(x) = \sum_{m=0}^{p-1} {}_m \exp(ik d_m^p$$

The second tool is a set of special results about the approximation of polynomials by plane waves which can be derived combining Lemma 3.10 and Proposition 3.9 in [9]. In that article, the estimates target a family of triangles and the unit square, here we need the estimates on the unit disk only.

Lemma 6.3. For odd p 5, $\hat{k} > 0$, and any $\hat{p}_1$ 2 $P_1(B_1)$, we can nd $\hat{v}_p$ 2 $PW_{p;\hat{k}}(B_1)$ such that

$$k\hat{p}_1 \quad \hat{v}_p k_{0;B_1} \quad C\hat{k}^2\, k\hat{p}_1 k_{0;B_1}\,; \tag{46}$$

$$j\hat{p}_1 \quad \hat{v}_p j_{1;B_1} \quad C(\hat{k}+1)\,\hat{k}^2\, k\hat{p}_1 k_{0;B_1}\,; \tag{47}$$

$$j\hat{v}_p j_{2;B_1} \quad C(\hat{k}+1)^2\hat{k}^2\, k\hat{p}_1 k_{0;B_1}\,: \tag{48}$$

Based on this lemma, we prove other auxiliary estimates.

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Lemma 6.4. Fix odd p 5. For every K 2 T

$$
\begin{aligned}
&C h_K^{\frac{1}{2} \; s} \; |\hat{u}|_{\frac{3}{2}+s;\hat{K}} + (h_K k + 1)^2 h_K^2 \, k^2 \, \|\hat{u}\|_{0;B_0} \\
&C \; |u|_{\frac{3}{2}+s;K} + (h_K k + 1)^2 h_K^{\frac{1}{2} \; s} k^2 \, \|u\|_{0;K} \; :
\end{aligned}
\tag{42}
$$

$\square$

and

$$\sum_{K\in\mathcal{T}_L\setminus\mathcal{L}_L^L} \|\kappa^{-1}\kappa\nabla(u-v_L)\cdot n\|^2_{0;\partial K} + \frac{\kappa h_{\max}}{h_K}\|u-v_L\|^2_{0;\partial K}$$

$$\overset{(41)}{\lesssim} \sum_{K\in\mathcal{T}_L\setminus\mathcal{L}_L^L} \left(1+\frac{1}{\kappa h_K}\right)\left[e^{-bq}+\frac{1+(\kappa h_K)^{q+1}}{c_0(q+1)^{\frac{q}{2}}}\right]^2\left(\|u\|_{L^\infty(K)}+h_K\|\nabla u\|_{L^\infty(K)}\right)^2$$

$$\lesssim \left\{ e^{-2bq}\sum_{K\in\mathcal{T}_L\setminus\mathcal{L}_L^L}\left(1+\frac{1}{\kappa h_K}\right) + \frac{1}{c_0(q+1)^{q}}\sum_{K\in\mathcal{T}_L\setminus\mathcal{L}_L^L}\left(\frac{1}{\kappa h_K}+(\kappa h_K)^{2q+2}\right)\right\}$$

$$\overset{(36),(34)}{\lesssim}$$

The proof of Theorem 6.5 shows that the rate $\breve{b}$ of exponential convergence of the Tre tz-DG method and the layer number threshold L only depend on: (i) the maximum number of elements per layer, which is bounded (see (36)); (ii) the regularity parameter s relative to the solution u; (iii) the mesh grading parameter ; (iv) the parameter b from Proposition 5.1 (and [15, Corollary 4.11]), which is the exponential convergence rate for the approximation of certain harmonic functions by harmonic polynomials.

Remark 6.6.

[11]

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